Lecture 2   Principal fibre bundles

We will now specialise to principal fibre bundles, so called because the typical fibre is a principally homogeneous space for a Lie group. (I realise that I may have just related two things you are unfamiliar with, so let's start with some definitions.)

7. Scholium   A **Lie group** consists of a manifold G which is also a group and such that group multiplication $G\times G \to G$, $(g,h)\mapsto gh$, and group inversion $G\to G$, $g\mapsto g^{-1}$, are smooth maps. If $g\in G$, we define diffeomorphisms $L_g : G\to G$, $h\mapsto gh$, and $R_g : G \to G$, $h\mapsto hg$, called **left** & **right** multiplication by g. A vector field $\xi\in\mathfrak{X}(G)$ is **left-invariant** if $(L_g)_*\xi = \xi$ $\forall g\in G$. In other words, ξ is left-invariant if $\xi(g) = (L_g)_*\xi(e)$, with $e\in G$ the identity. Similarly, ξ is **right-invariant** if $(R_g)_*\xi=\xi$ $\forall g\in G$. The Lie bracket of two left-invariant vector fields is left-invariant. The vector space of left-invariant vector fields defines the **Lie algebra** $\mathfrak{g}$ of G. Since a LIVF is uniquely determined by its value at the identity, $\mathfrak{g}\cong T_eG$ as a vector space, but we can transport the Lie bracket from $\mathfrak{g}$ to T_eG in such a way that $\mathfrak{g}\cong T_eG$ as a Lie algebra. We will identify $\mathfrak{g}$ with T_eG at times. Under that identification, the maps (as $g\in G$) $(L_{g^{-1}})_* : T_gG \to T_eG = \mathfrak{g}$ define a one-form θ with values in $\mathfrak{g}$, called the **left-invariant Maurer–Cartan one-form**. If ξ is a LIVF, $\theta(\xi) = \xi(e)$. By definition, θ is **left-invariant**: $L_g^*\theta = \theta$ $\forall g\in G$. It satisfies the **structure equation** $d\theta = -\frac{1}{2}[\theta,\theta]$ where $[\theta,\theta]$ is both the $\wedge$ of one-forms and the Lie bracket. In other words, $d\theta(\xi,\eta) = -[\theta(\xi),\theta(\eta)]$ for vector fields $\xi,\eta\in\mathfrak{X}(G)$.
If G is a matrix Lie group, $\theta_g = g^{-1}dg$ from where left-invariance and the structure equation follows.
Every $g\in G$ defines a diffeomorphism $L_g\circ R_{g^{-1}} : G\to G$, $h\mapsto ghg^{-1}$. Since $geg^{-1}=e$, its derivative belongs to $GL(T_eG) = GL(\mathfrak{g})$. This defines the **adjoint representation** of G on $\mathfrak{g}$: $Ad_g = (L_g)_*\circ(R_g^{-1})_*$
Notice that $R_g^*\theta = Ad_{g^{-1}}\circ\theta$ (Pf. $R_g^*\theta_{hg} = \theta_{hg}\circ(R_g)_* = (L_{(hg)^{-1}})_*\circ(R_g)_* = (L_{g^{-1}})_*\circ(L_{h^{-1}})_*\circ(R_g)_* = (L_{g^{-1}})_*\circ(R_g)_*\circ(L_{h^{-1}})_* = Ad_{g^{-1}}\circ\theta_h$)

A **left action** of a Lie group G on a manifold M is a smooth map $G \times M \longrightarrow M$, denoted $(g,a) \mapsto ga$, satisfying the axioms $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ and $e \cdot a = a$ $\forall a \in M$, $\forall g_1, g_2 \in G$. (The last condition follows if e act via a diffeomorphism.)

Similarly, one defines a **right action** $M \times G \longrightarrow M$, $(a,g) \mapsto a \cdot g$ to be a smooth map such that $\forall g_1, g_2 \in G$ and $a \in M$, $(a \cdot g_1) \cdot g_2 = a \cdot (g_1 g_2)$ and $e \cdot a = a$. There is no real difference between left and right actions: if G acts on M on the right, we can define a left action by $g \cdot a := a \cdot g^{-1}$ and viceversa.

An action of G on M is said to be **transitive** if the G-orbit of any point is all of M; equivalently if given any two points in M, there is some group element taking one to the other. The action is said to be **free** if the only element which fixes any point is the identity. A **G-torsor** is a manifold M on which G acts freely and transitively. It follows that choosing any point in a G-torsor M defines a diffeomorphism $G \cong M$. So a G-torsor is like a Lie group where we have forgotten the identity. A vector space V is an abelian Lie group and a V-torsor is just an affine space modelled on V. G-torsors are also called **principally homogeneous** G-spaces.

8. Definition Let G be a Lie group. A **principal G-bundle** is a fibre bundle $P \xrightarrow{\pi} M$ together with a smooth right G-action $P \times G \longrightarrow P$, $(p,g) \mapsto r_g(p) = pg$, which preserves the fibres ($\pi(pg) = \pi(p)$ $\forall p \in P$, $g \in G$) and acts freely and transitively on them. It follows that the fibres are the G-orbits and hence $M = P/G$. The condition of local triviality now says that the local trivialisations $\pi^{-1}(U) \xrightarrow[\cong]{\varphi} U \times G$ are G-equivariant, where $\varphi(p) = (\pi(p), \gamma(p))$ $\exists$ G-equivariant smooth map $\gamma: \pi^{-1}(U) \longrightarrow G$ which is a fibrewise diffeomorphism. Equivariance means $\gamma(pg) = \gamma(p) g$.

A principal G-bundle $P \xrightarrow{\pi} M$ is **trivial** if $\exists$ a G-equivariant diffeomorphism $P \xrightarrow{\psi} M \times G$.

9. Proposition A principal G-bundle $P \xrightarrow{\pi} M$ admits a section if and only if it is trivial.

Proof If $P \xrightarrow{\pi} M$ is trivial, $\psi: P \longrightarrow M \times G$, we define a section $s: M \to P$ by $s(a) = \psi^{-1}(a,e)$.

If $s: M \longrightarrow P$ is a section, we define $\psi: P \longrightarrow M \times G$ by $\psi(p) = (\pi(p), \chi(p))$ where $\chi(p) \in G$ is uniquely determined by $p = s(\pi(p))\chi(p)$. Notice that $\chi(pg) = \chi(p) g$, since $s(\pi(p))\chi(p) g = s(\pi(pg))\chi(pg) = pg$. ■

10. Example Let G be a Lie group and $H \subset G$ a closed Lie subgroup. Then $G \xrightarrow{\pi} G/H$ is a principal H-bundle. Therefore homogeneous spaces are examples of principal bundles.

Since PFBs are locally trivial, they admit local sections. Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be a trivialising atlas for $G \to P \xrightarrow{\pi} M$. The canonical local sections $s_\alpha : U_\alpha \to \pi^{-1}U_\alpha$ are given by $s_\alpha(a) := \varphi_\alpha^{-1}(a, e)$. On $U_{\alpha\beta}$, we have sections s_α and s_β. How are they related?

Let

$$U_{\alpha\beta} \times G \xleftarrow{\varphi_\beta} \pi^{-1}U_{\alpha\beta} \xrightarrow{\varphi_\alpha} U_{\alpha\beta} \times G \qquad \varphi_\alpha(p) = (\pi(p), g_\alpha(p)) \quad g_\alpha : U_\alpha \to G \text{ s.t. } g_\alpha(p \cdot g) = g_\alpha(p) g$$

Let $p \in \pi^{-1}U_{\alpha\beta}$. Then

$$(\pi(p), g_\alpha(p)) = \varphi_\alpha(p) = (\varphi_\alpha \circ \varphi_\beta^{-1} \circ \varphi_\beta)(p) = (\varphi_\alpha \circ \varphi_\beta^{-1})(\pi(p), g_\beta(p))$$

$$\therefore \quad (\pi(p), \underbrace{g_\alpha(p) g_\beta^{-1}(p)}_{=: \hat{g}_{\alpha\beta}(p)} g_\beta(p)) = (\varphi_\alpha \circ \varphi_\beta^{-1})(\pi(p), g_\beta(p))$$

Notice that $\hat{g}_{\alpha\beta}(p \cdot g) := g_\alpha(p \cdot g)\, g_\beta(p \cdot g)^{-1} = g_\alpha(p) g g^{-1} g_\beta(p)^{-1} = \hat{g}_{\alpha\beta}(p)$, so it's constant on the fibres and hence

$$\hat{g}_{\alpha\beta} = \pi^* g_{\alpha\beta} \qquad \exists\, g_{\alpha\beta} : U_{\alpha\beta} \to G \qquad \text{and} \qquad (\varphi_\alpha \circ \varphi_\beta^{-1})(a, g) = (a, g_{\alpha\beta}(a) g)$$

It follows that $\{g_{\alpha\beta} : U_{\alpha\beta} \to G\}$ obeys the cocycle conditions. Then $\{L \circ g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Diff}(G)\}$ are the transition functions of the PFB, although one often refers to $\{g_{\alpha\beta}\}$ as the transition functions.

Finally we can answer the question: how are the canonical local sections related on overlaps?
Notice that $g_\alpha \circ s_\alpha$ is the constant map $U_\alpha \to G$ sending $a \mapsto e$ $\forall a \in U_\alpha$. Then,
$g_\alpha(p) = g_{\alpha\beta}(\pi(p))\, g_\beta(p)$, hence for $p = s_\beta(a)$, $\quad g_\alpha(s_\beta(a)) = g_{\alpha\beta}(a) = g_\alpha(s_\alpha(a))\, g_{\alpha\beta}(a) = g_\alpha(s_\alpha(a) g_{\alpha\beta}(a))$
Since g_α is a diffeomorphism, $\boxed{s_\beta(a) = s_\alpha(a)\, g_{\alpha\beta}(a)}$ $\forall a \in U_{\alpha\beta}$.